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Technical Notes

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Extrapolation Procedures for the Time-Dependent Navier-Stokes Equations

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Introduction

When solving partial differential equations in unbounded domains, the most common procedure is to introduce artificial boundaries Γ. If the computational domain Ω is chosen large enough, one can usually get sufficient accuracy by using well-posed boundary conditions of standard type with data obtained from the state at infinity; see, for example, Refs. 1 and 2.

For many types of equations it is also possible to use smaller computational domains by substituting the true equations by simpler ones in the exterior domain. In this way one can use the form of the solution of the simplified equations to construct boundary conditions for the true equations in the inner domain. Such procedures usually lead to nonlocal conditions; see, for example, Refs. 3–7.

In many cases it is impossible to use methods based on simplified models outside Γ. For example, the geometry may be nontrivial, not permitting any simplifications, or the flowfield might be highly nonlinear. The question then is whether or not one can construct boundary operators L giving accurate solutions u in Ω despite the fact that the true data g in $Lu = g$ are not known.

A very common procedure is to extrapolate the solution from gridpoints near the boundary to the boundary itself. Linear extrapolation is an approximation of the condition $\partial^2 u/\partial n^2 = 0$, where n is normal to the boundary. For first- and second-order systems like the Euler and Navier-Stokes equations, one cannot in general expect accurate solutions. In Ref. 8, the steady Navier-Stokes equations were analyzed, and it was shown that extrapolation methods can be used with good results if there are large gradients tangential to the boundary. In this Note, we claim that extrapolation methods also can be used for the time-dependent problem.

Analysis

Let us consider the flowfield depicted in Fig. 1. We wish to solve the initial boundary value problem in Eqs. (1–3) in the unbounded domain Ω_∞,

$$u_t = P(u)u + F(x,t), \quad x \in \Omega_\infty, \quad t \geq 0 \qquad (1)$$

$$u = f(x), \quad x \in \Omega_\infty, \quad t = 0 \qquad (2)$$

$$L_s u = g_s(t), \quad x \in \Gamma_s, \quad t \geq 0 \qquad (3)$$

where P is a general differential operator and L_s the boundary operator; F, f, and g_s are the forcing function, the initial function, and the boundary data, respectively. In practice we introduce artificial boundaries and try to solve the problem of Eqs. (4–8) in the bounded domain Ω:

$$v_t = P(v)v + F(x,t), \quad x \in \Omega, \quad t \geq 0 \qquad (4)$$

$$v = f(x), \quad x \in \Omega, \quad t = 0 \qquad (5)$$

$$L_s v = g_s(t), \quad x \in \Gamma_s, \quad t \geq 0 \qquad (6)$$

$$L_O v = g_O(t), \quad x \in \Gamma_O, \quad t \geq 0 \qquad (7)$$

$$L_I v = g_I(t), \quad x \in \Gamma_I, \quad t \geq 0 \qquad (8)$$

Note that if P denotes the Navier-Stokes operator with a no-slip condition at the solid wall Γ_s, then we will have subsonic outflow at the part of Γ_O that is close to Γ_s.

The difference or error $w = u - v$ satisfies the linearized problem of Eqs. (9–13):

$$w_t = P(u)w, \quad x \in \Omega, \quad t \geq 0 \qquad (9)$$

$$w = 0, \quad x \in \Omega, \quad t = 0 \qquad (10)$$

$$L_s w = 0, \quad x \in \Gamma_s, \quad t \geq 0 \qquad (11)$$

$$L_O w = \tilde{g}_O(t), \quad x \in \Gamma_O, \quad t \geq 0 \qquad (12)$$

$$L_I w = \tilde{g}_I(t), \quad x \in \Gamma_I, \quad t \geq 0 \qquad (13)$$

Normally one knows boundary data with good accuracy at inflow boundaries (Γ_I), whereas knowledge of data at outflow boundaries (Γ_O), often is lacking. In other words, one often has $\tilde{g}_I = g_I - L_I w \approx 0$ and $\tilde{g}_O = g_O - L_O w = \mathcal{O}(1)$. In this Note, we will investigate outflow boundary operators L_o of extrapolation type where the error in the boundary data is of order one.

From now on we consider the Navier-Stokes equations. The dependent variables and parameters ρ, u, v, T, p, c, M_∞, μ, λ, κ, Pr, γ, and ϵ are, respectively, the density, x and y compo-

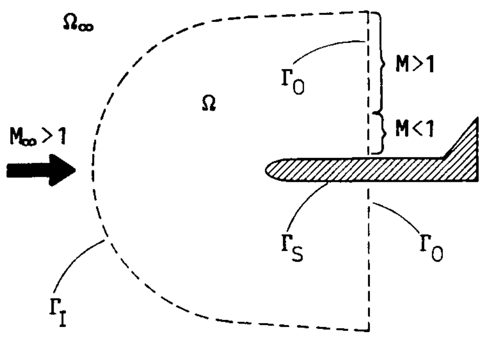

Fig. 1 Supersonic flow past a vehicle.

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nents of the velocity, the temperature, the pressure, the speed of sound, the freestream Mach number, the shear and second viscosity, the coefficient of heat conduction, the Prandtl number, the ratio of specific heats, and the inverse Reynolds number.

The equations are linearized around a basic flow with small streamwise x gradients and large transversal y gradients. To be more precise, we make the following assumption.

Assumption 1

$$\bar{\rho}_x,\ \bar{u}_x,\ \bar{v}_x,\ \bar{T}_x,\ \bar{v}_y \equiv 0, \quad \bar{\rho}_y,\ \bar{u}_y,\ \bar{T}_y = \mathcal{O}(\epsilon^{-q}) \tag{14}$$

where $0 < q < 1$.

Note that the assumption, Eq. (14), is a good approximation of the conditions in a flowfield close to an isothermal wall, in which case $q \approx 1/2$. If we consider perturbations that are independent of y, we obtain the linearized unsteady one-dimensional equation for a quarter space:

$$\phi_t = \bar{P}\phi, \quad x \le 0, \quad t \ge 0 \tag{15}$$

$$\phi = 0, \quad x \le 0, \quad t = 0 \tag{16}$$

$$\partial^r \phi^{(2)}/\partial x^r = g(t), \quad x = 0, \quad t \ge 0 \tag{17}$$

$$\phi \to 0, \quad x \to -\infty, \quad t \ge 0 \tag{18}$$

where

$$\bar{P} = -\epsilon^{-q}\bar{B} - \bar{A}\partial/\partial x + \epsilon\bar{C}\partial^2/\partial x^2$$

$$\phi = (\rho, u, v, T)^T, \qquad \phi^{(2)} = (u, v, T)^T$$

$$g = (g_2, g_3, g_4)^T, \qquad r \ge 0, \quad \bar{u} > 0$$

and

$$\bar{A} = \begin{bmatrix} \bar{u} & \bar{\rho} & 0 & 0 \\ \bar{c}^2/(\gamma\bar{\rho}) & \bar{u} & 0 & 1/(\gamma M_\infty^2) \\ 0 & 0 & \bar{u} & 0 \\ 0 & (\gamma-1)\bar{c}^2 M_\infty^2 & 0 & \bar{u} \end{bmatrix} \tag{19}$$

$$\bar{B} = \begin{bmatrix} 0 & 0 & \bar{\rho}_y & 0 \\ 0 & 0 & \bar{u}_y & 0 \\ -\bar{c}^2\bar{\rho}_y/(\gamma\bar{\rho}^2) & 0 & 0 & \bar{\rho}_y/(\gamma M_\infty^2 \bar{\rho}) \\ 0 & 0 & \bar{T}_y & 0 \end{bmatrix} \tag{20}$$

$$\bar{C} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & (\bar{\lambda} + 2\bar{\mu})/\bar{\rho} & 0 & 0 \\ 0 & 0 & \bar{\mu}/\bar{\rho} & 0 \\ 0 & 0 & 0 & \gamma\bar{k}/\bar{\rho}Pr \end{bmatrix} \tag{21}$$

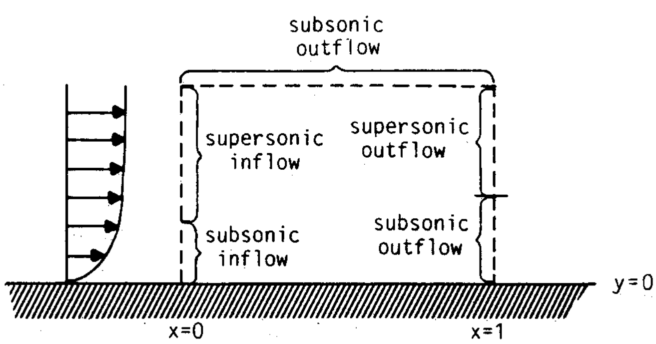

Fig. 2 Geometry definition.

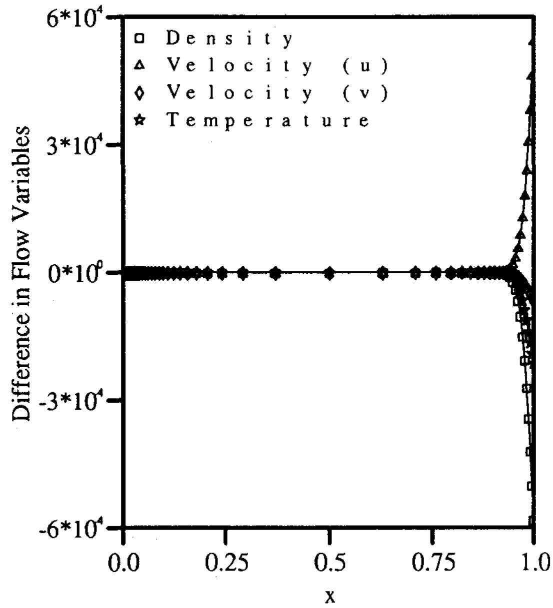

Fig. 3 Difference distribution, $r = 2$, $\epsilon = 10^{-4}$.

The matrices $\bar{A}, \bar{B}$, and $\bar{C}$ are now of order $\mathcal{O}(1)$. If $\bar{u} < \bar{c}$, we have subsonic flow, and if $\bar{u} > \bar{c}$, we have supersonic flow.

Note the presence of the large term $\epsilon^{-q}\bar{B}\phi$ in Eq. (15). Nondifferentiated terms are usually not included in a linearized equation. However, the presence of this term is the reason why we obtain an accurate solution in the subsonic case. We make the following proposition.

Proposition 1

The solution of the problem of Eqs. (15–18) satisfies

$$\int_0^T |\phi|^2 e^{-2\eta t} \ \mathrm{d}t \le c_1 \epsilon^{2r} e^{2\alpha_1 x/\epsilon} \int_0^T |g|^2 e^{-2\eta t} \ \mathrm{d}t \tag{22}$$

$$\int_0^T |\phi|^2 e^{-2\eta t} \ \mathrm{d}t \le c_2 \epsilon^{2rq} e^{2\alpha_2 x/\epsilon^q} \int_0^T |g|^2 e^{-2\eta t} \ \mathrm{d}t \tag{23}$$

if $\bar{u} < \bar{c}$. The constants $c_1, c_2, \alpha_1, \alpha_2, \eta = \mathcal{O}(1) > 0$.

The proof of Proposition 1 will be given in a forthcoming paper; here we will indicate its validity by numerical experiments. Proposition 1 means that 1) the accuracy of the solution increases with the order of the derivative boundary condition and 2) the error in the solution decays exponentially fast away from the boundary. Note that we have an accurate solution also in the subsonic case if Assumption 1 holds and $r > 0$.

Numerical Experiments

By making computations using the nonlinear Navier-Stokes equations, we can check whether the theoretical conclusion drawn from the simplified problem of Eqs. (15–18) agrees with the results obtained in practice. To obtain the numerical solutions, we used a centered finite-volume discretization in space and the classical fourth-order Runge-Kutta method in time.

We consider the flow over a flat plate; see Fig. 2. The boundary conditions at the inflow boundary, the solid wall, and the upper boundary were fixed. The boundary conditions used at the solid wall were $u = 0$, $v = 0$, $T = T_\infty$. At the subsonic inflow boundary, we used $u + 2c/(\gamma - 1) = h_1$, $T\rho^{1-\gamma} = h_2$, $\theta u_x - 2(k/Pr)c_x = h_3$, and $v = h_4$. The supersonic inflow boundary conditions were $\rho = r_1$, $u = r_2$, $v = r_3$, and $T = r_4$. At the upper boundary, we used $u_y = 0$, $v_y = 0$, and $T_y = 0$. We have accurate data at the upper boundary, and therefore we can use derivative boundary conditions. The functions $h_i(y)$

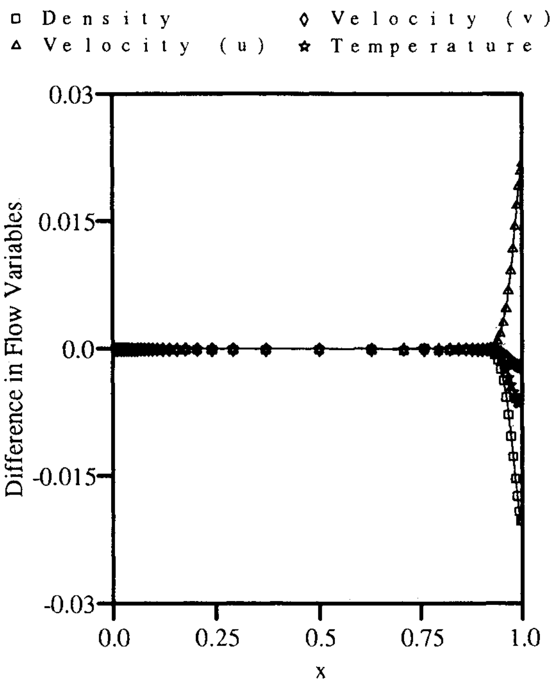

Fig. 4 Difference distribution, $r = 1$, $\epsilon = 10^{-4}$.

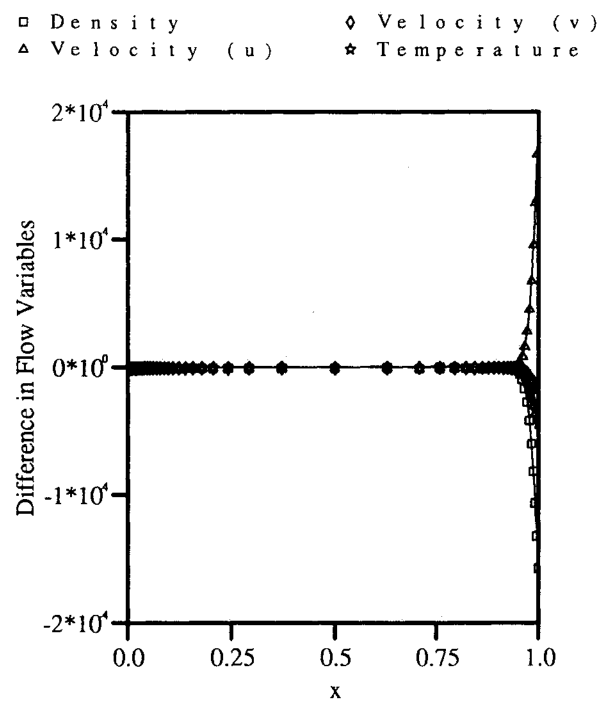

Fig. 5 Difference distribution, $r = 2$, $\epsilon = 10^{-5}$.

and $r_i(y)$ $i = 1$, 2, 3, and 4 were obtained by solving the compressible boundary-layer equations for the flow over a flat plate.

The boundary conditions at the outflow boundary were

$$\frac{\partial^r u}{\partial x^r} = g_2, \quad \frac{\partial^r v}{\partial x^r} = 0, \quad \frac{\partial^r T}{\partial x^r} = 0 \tag{24}$$

where $g_2 = 0$ or $g_2 = \sin\,[\Theta(t)]$, $\Theta(t) = 4\pi t$. The computations were made at $M_\infty = 2$. We have computed the flowfield for $\epsilon = 10^{-5}$ and $\epsilon = 10^{-4}$ using linear extrapolation ($r = 2$) at the outflow boundary. For $\epsilon = 10^{-4}$ we have also computed the flow using constant extrapolation ($r = 1$). The time dependency of the flow is modeled in the following way. First, a steady solution using $g_2 = 0$ was computed. Second, the steady solution was used as the initial solution and advanced in time using $g_2 = \sin[\Theta(t)]$. Finally, the difference between the solution at a given time and the initial solution was studied. The differences in all figures are evaluated at the y value where the local Mach number is approximately 0.55.

Results and Discussion

The error estimates in Proposition 1 indicate that the error close to $x = 1$, due to the subsonic flow, should have the form $\Delta(\epsilon,r) \approx$ const $\times \epsilon^{rq}$. If we assume that $q = 1/2$, we get the following theoretical estimates:

$$\Delta_1 \approx \text{const} \times 10^{-4} \tag{25}$$

$$\Delta_2 \approx \text{const} \times 10^{-2} \tag{26}$$

$$\Delta_3 \approx \text{const} \times 10^{-5} \tag{27}$$

In Figs. 3-5 the three different cases are shown at $\Theta = \pi$. The amplitudes at the outflow boundary are

$$\Delta_1 \approx 5.5 \times 10^{-4} \tag{28}$$

$$\Delta_2 \approx 2.2 \times 10^{-2} \tag{29}$$

$$\Delta_3 \approx 1.7 \times 10^{-4} \tag{30}$$

The theoretical estimates of Eqs. (25-27) and the practical results of Eqs. (28-30) agree quite well. The same degree of agreement is obtained also for other values of Θ. Furthermore, the boundary-layer character of the difference between the solutions predicted in Proposition 1 is clearly seen.

Conclusions

It is proposed that extrapolation procedures at subsonic artificial outflow boundaries can be used for the time-dependent compressible Navier-Stokes equations if sufficiently large transverse gradients are present in the flowfield. The proposition is confirmed by numerical calculations.

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